

Random quantum measurements are almost maximally incompatible

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QP47, Monastir, October 27th 2026



Quantum measurements



Quantum measurements



state

$$\rho \in \mathcal{M}_d(\mathbb{C})$$
$$\rho \succeq 0, \quad \text{Tr } \rho = 1$$

measurement (POVM)

$$A_1, \dots, A_k \in \mathcal{M}_d(\mathbb{C})$$
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Born rule

$$\mathbb{P}(a) = \text{Tr}[\rho A_a]$$

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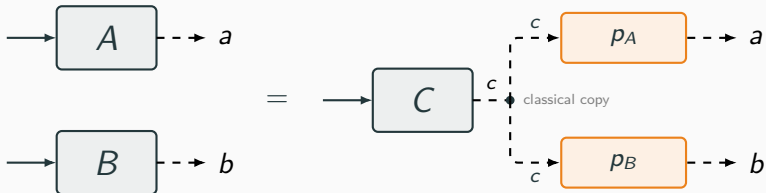
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- A **projective** measurement (PVM): all A_a are orthogonal projections, e.g. measuring in a basis, $A_a = |e_a\rangle\langle e_a|$, $a \in \{1, \dots, d\}$

Compatible measurements

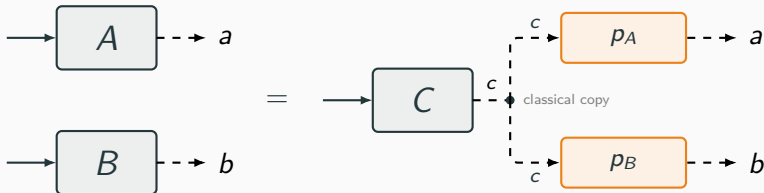
Two measurement devices on $\mathbb{C}^d$ are **compatible**, $A \oslash B$, if they can both be simulated by **one** device C , followed by **classical post-processing**:



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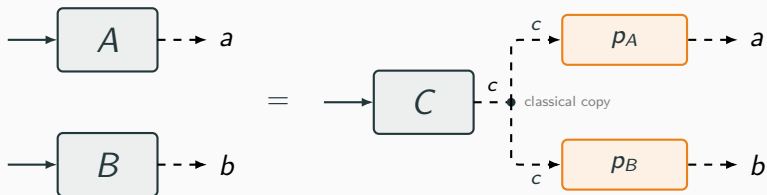
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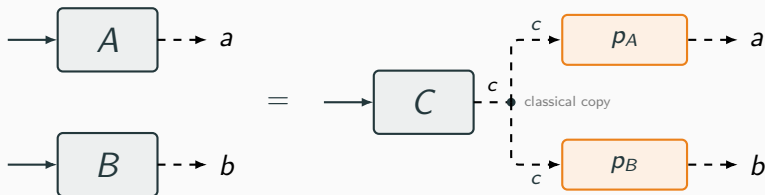
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- Incompatibility is a quantum **resource**: Bell nonlocality, steering, etc.¹

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Compatibility via marginals

$A \oslash B \iff$ there is a joint POVM $J = (J_{ab})$ having marginals A and B

$$\forall a \quad A_a = \sum_b J_{ab}$$

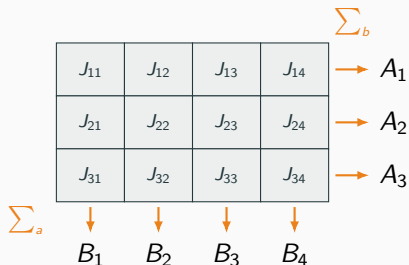
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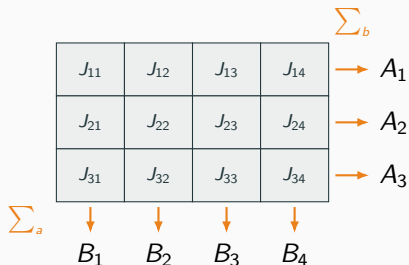
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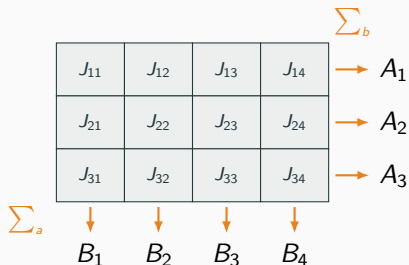
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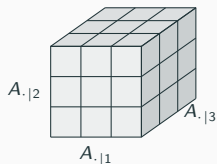
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g measurements $A_{\cdot|1}, \dots, A_{\cdot|g}$ ($A_{a|x} \rightarrow$ outcome a of measurement x) are compatible iff there is a joint POVM $(J_{\mathbf{b}})$, $\mathbf{b} \in [k_1] \times \dots \times [k_g]$, with

$$\sum_{\mathbf{b} : b_x = a} J_{\mathbf{b}} = A_{a|x}$$

Examples

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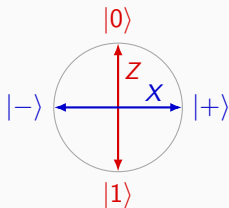
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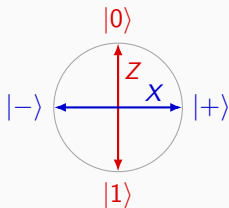
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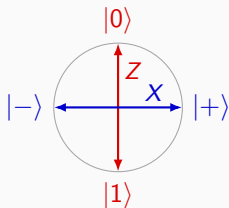
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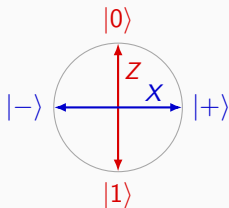
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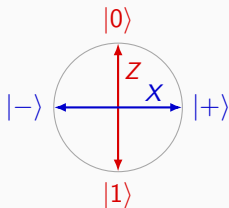
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• If either A or B is projective, then: $A \oslash B \iff [A_a, B_b] = 0, \forall a, b$

Noisy measurements

Mix A with the **trivial** measurement (uniformly random outcome):

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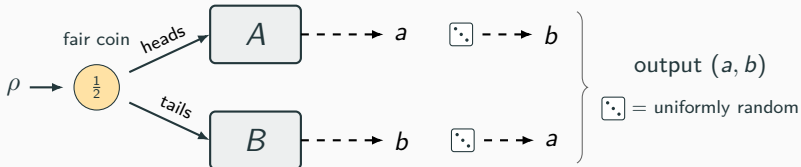
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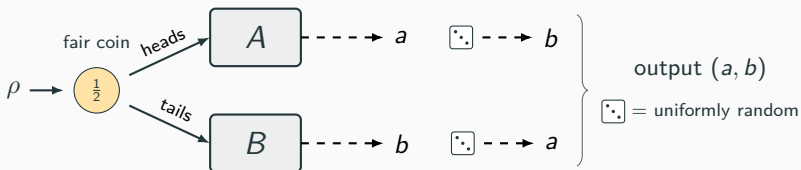


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$$J_{ab} := \frac{1}{2l} A_a + \frac{1}{2k} B_b \quad \Rightarrow \quad \sum_b J_{ab} = \frac{1}{2} A_a + \frac{1}{2} \frac{1}{k} = A_a^{(1/2)}$$

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- the coin flip of the previous slide: $\tau(d, 2, (k_1, k_2)) \geq 1/2$
- least noise making **any** g measurements compatible:
 $\rightsquigarrow \tau$ is a **fundamental constant of nature**

Maximally incompatible measurements

- A fair g -sided die instead of a coin: $\tau(d, g, (k_1, \dots, k_g)) \geq 1/g$

¹A. Bluhm, I. Nechita, [Joint measurability of quantum effects and the matrix diamond](#), J. Math. Phys. **59**, 112202 (2018)

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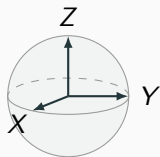
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anticommuting = orthogonal

g	optimal $F_1, \dots, F_g$	d
2	X, Z	2
3	X, Y, Z	2
4	$X \otimes X, X \otimes Y, X \otimes Z, Y \otimes I$	4
5	$X \otimes X, X \otimes Y, X \otimes Z, Y \otimes I, Z \otimes I$	4

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Random measurements

For large enough dimension d , the most incompatible dichotomic measurements have $\tau = 1/\sqrt{g}$.

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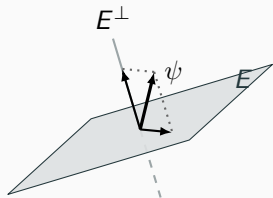
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Random dichotomic **projective** measurement: $A = (P_E, P_{E^\perp})$ with $E \subset \mathbb{C}^d$ a **uniformly random** subspace of dimension $\lfloor \alpha d \rfloor$, $\alpha \in (0, 1)$



outcome YES $\leftrightarrow E$, NO $\leftrightarrow E^\perp$

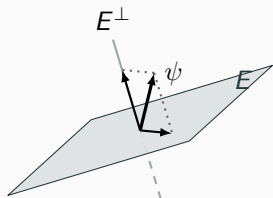
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How to sample: U Haar-random unitary

- Let $|u_1\rangle, \dots, |u_d\rangle$ be the columns of U
- Let $E = \text{span}\{|u_1\rangle, \dots, |u_{\lfloor \alpha d \rfloor}\rangle\}$
- For $E^\perp$, take the remaining columns of U

$$P_E = \sum_{i=1}^{\lfloor \alpha d \rfloor} |u_i\rangle\langle u_i| \quad \text{and} \quad P_{E^\perp} = \sum_{i=\lfloor \alpha d \rfloor+1}^d |u_i\rangle\langle u_i|$$

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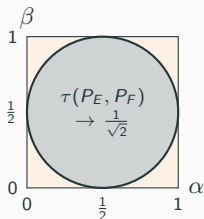
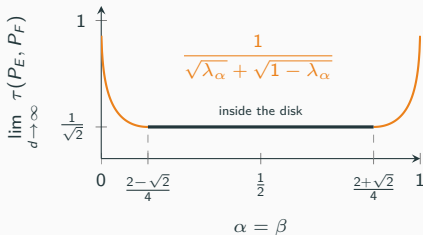
Main results: two random measurements

$E, F \subset \mathbb{C}^d$ independent random subspaces, $\dim E \sim \alpha d$, $\dim F \sim \beta d$

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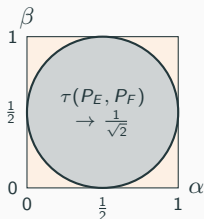
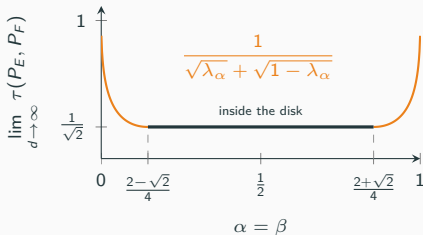
inside the disk: **maximally incompatible**

less incompatible outside the disk; $\lambda_\alpha = 4\alpha(1 - \alpha)$

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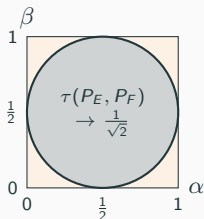
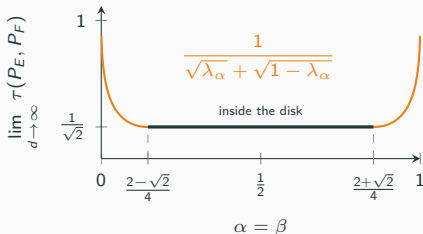
inside the disk: **maximally incompatible**

less incompatible outside the disk; $\lambda_\alpha = 4\alpha(1 - \alpha)$

- **upper bound** (incompatibility): a hidden qubit $\rightsquigarrow$ next slide

Main results: two random measurements

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- **upper bound** (incompatibility): a hidden qubit $\rightsquigarrow$ next slide
- **lower bound** (compatibility): the **Jordan product criterion**¹

$$A_a B_b + B_b A_a \succeq 0 \quad \forall a, b \implies A \oslash B \quad \left(\text{take } J_{ab} := \frac{1}{2}(A_a B_b + B_b A_a) \succeq 0 \right)$$

applied to $P_E^{(t)}, P_F^{(t)}$: **tight** as $d \rightarrow \infty$, see next slide

¹T. Heinosaari, *A simple sufficient condition for the coexistence of quantum effects*, J. Phys. A **46**, 152002 (2013)

Proof sketch ($g = 2$)

Idea: a hidden qubit $\mathbb{C}^2 \subset \mathbb{C}^d$ on which P_E acts as Z and P_F acts as X

¹P. R. Halmos, *Two subspaces*, Trans. Amer. Math. Soc. **144**, 381–389 (1969)

²G. Aubrun, *Principal angles between random subspaces and polynomials in two free projections*, Confluentes Math. **13**(2), 3–10 (2021)

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Principal angles between two subspaces E, F

$\mathbb{C}^d$ splits¹ into blocks of dimension ≤ 2 , invariant under both P_E and P_F ; on each 2-dimensional block

$$P_E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_F = \begin{pmatrix} \cos^2 \theta_i & \cos \theta_i \sin \theta_i \\ \cos \theta_i \sin \theta_i & \sin^2 \theta_i \end{pmatrix}, \quad \theta_i \in (0, \frac{\pi}{2})$$

$\cos \theta_i =$ singular values of $V_E^* V_F$ (V_E, V_F : orthonormal bases) $\theta_i = \frac{\pi}{4} \rightsquigarrow Z, X$

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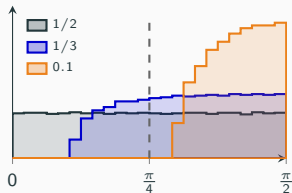
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principal angles θ_i ($d = 4000$, $\alpha = \beta$)



- random E, F : the angles fill a limit law whose support contains $\frac{\pi}{4}$ iff (α, β) is in the disk² $\rightsquigarrow$ some $\theta_i \approx \frac{\pi}{4}$

- compression increases compatibility:

$$\tau(P_E, P_F) \leq \tau(\text{block } i) \approx \tau(Z, X) = \frac{1}{\sqrt{2}}$$

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Main results: g random measurements

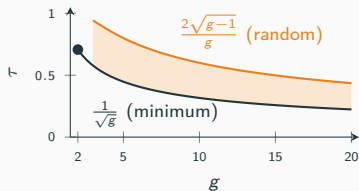
$P_1, \dots, P_g$ independent random projections of rank $d/2$, $A_x := 2P_x - I$

¹C. Carmeli, T. Heinosaari, A. Toigo, [Quantum incompatibility witnesses](#), Phys. Rev. Lett. **122**, 130402 (2019)

²A. Bluhm, I. Nechita, [A tensor norm approach to quantum compatibility](#), J. Math. Phys. **63**, 062201 (2022)

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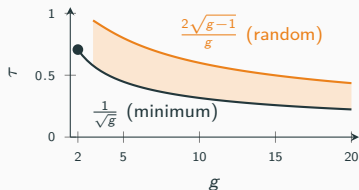
- $g \leq 3d$: $\tau \leq \frac{31}{\sqrt{g}}$ w.p. $\geq 1 - e^{-2d}$
 $\rightsquigarrow \tau(d, g, (2, \dots, 2)) = \Theta(\frac{1}{\sqrt{g}})$

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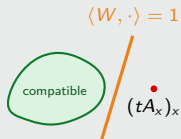
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Proof idea: incompatibility witnesses



- **witness** W : $\langle W, \cdot \rangle \leq 1$ on compatible tuples¹;
dichotomic² case: $\sum_x \varepsilon_x W_x \leq \rho \quad \forall \varepsilon \in \{\pm 1\}^g$
- choose $W_x \propto A_x$, $\rho = I/d$; **free probability**
(Kesten–McKay law): $\left\| \sum_x \varepsilon_x A_x \right\| \rightarrow 2\sqrt{g-1}$

¹C. Carmeli, T. Heinosaari, A. Toigo, [Quantum incompatibility witnesses](#), Phys. Rev. Lett. **122**, 130402 (2019)

²A. Bluhm, I. Nechita, [A tensor norm approach to quantum compatibility](#), J. Math. Phys. **63**, 062201 (2022)

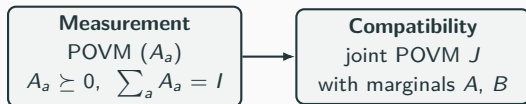
Take-home slide

Measurement

POVM (A_a)

$$A_a \succeq 0, \quad \sum_a A_a = I$$

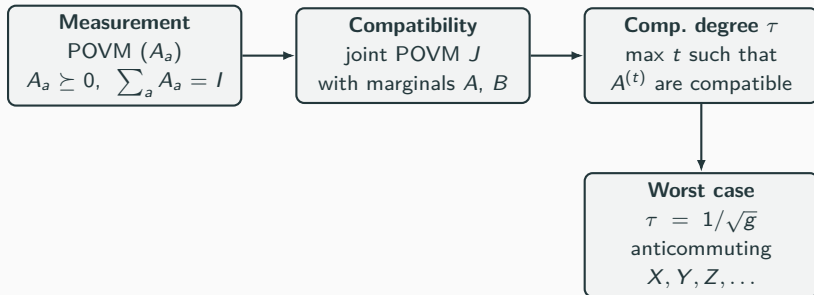
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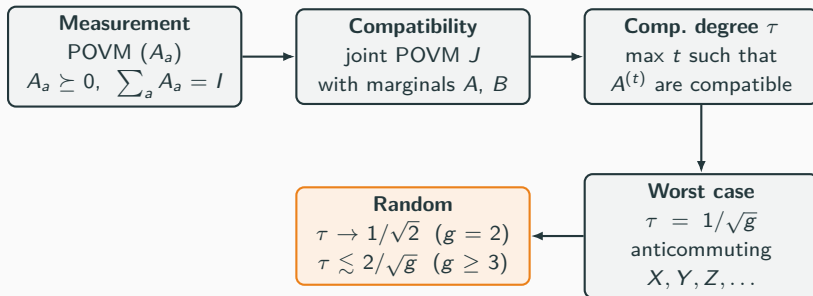
Take-home slide



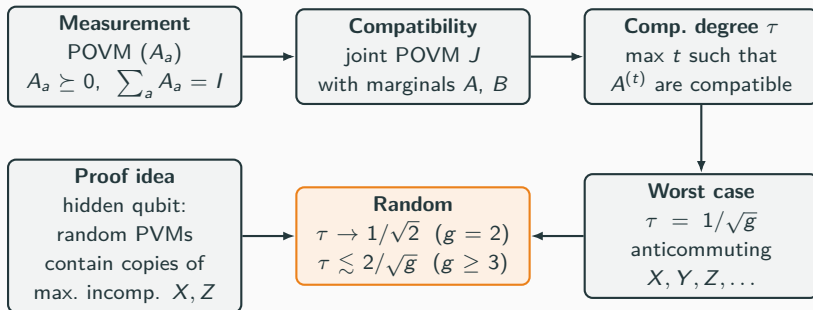
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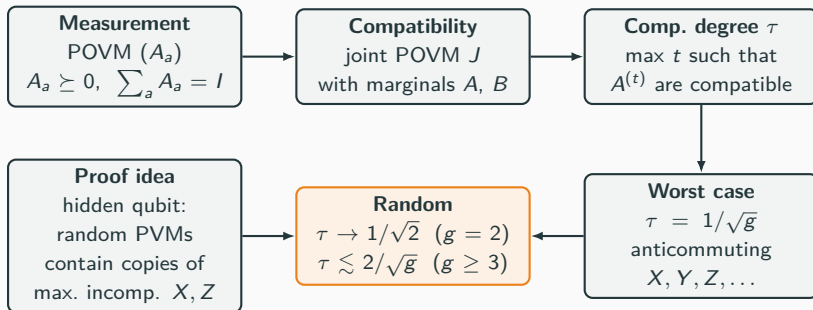
Take-home slide



Take-home slide



Take-home slide



Random measurements are almost maximally incompatible

A gap between maximal and typical?

Random measurements seem **not quite** maximally incompatible. The situation is somewhat similar to that of **entanglement**.

Consider bipartite pure states $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$. Their entanglement is measured by the **entanglement entropy**

$$S(\psi) = -\text{Tr } \rho \log \rho \quad \text{with} \quad \rho = \text{Tr}_2 |\psi\rangle\langle\psi|$$

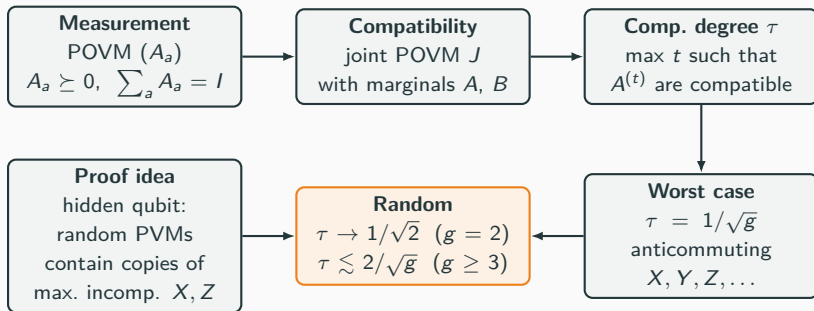
The maximally entangled state $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle$ has $S(\Omega) = \log d$

A random $|\varphi\rangle$ (uniform on the sphere of $\mathbb{C}^{d^2}$) has

$$S(\varphi) \sim \log d - \frac{1}{2} \quad \text{as } d \rightarrow \infty$$

Random states are **almost** maximally entangled: the gap $\frac{1}{2}$ does not grow with the Hilbert space dimension d

Take-home slide



Random measurements are almost maximally incompatible

Open questions

- exact asymptotic τ of $g \geq 3$ random dichotomic PVMs
- random **non-projective** measurements: close the compatible/incompatible gap